

Variational representations for N -cyclically monotone vector fields

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Abstract

Given a domain Ω in \mathbb{R}^d , we show that a bounded measurable vector field $u : \Omega \rightarrow \mathbb{R}^d$ is N -cyclically monotone for some $N \geq 2$, if and only if there exists a Hamiltonian H , which is concave in the first variable, convex in the last $(N - 1)$ variables such that

$$(-u(x), 0, \dots, 0, u(x)) \in \partial H(x, x, \dots, x) \text{ for } x \in \Omega.$$

Moreover, H is N -subsymmetric, meaning that $\sum_{i=0}^{N-1} H(\sigma^i \mathbf{x}) \leq 0$ for all $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$, σ being the cyclic permutation on \mathbb{R}^d defined by $\sigma(x_1, x_2, \dots, x_N) = (x_2, x_3, \dots, x_N, x_1)$. Furthermore, H is N -symmetric in the following sense

$$H(x_1, x_2, \dots, x_N) + H_{2, \dots, N-1}(x_1, x_2, \dots, x_N) = 0,$$

where $H_{2, \dots, N}$ is the concavification of the function $K(\mathbf{x}) = \sum_{i=1}^{N-1} H(\sigma^i \mathbf{x})$ with respect to the last $N - 1$ variables. This can be seen as an extension of a theorem of Krauss, which associates to any monotone operator, a concave-convex anti-symmetric saddle function.

1 Introduction

Given a domain Ω in \mathbb{R}^d , recall that a single-valued map u from Ω to \mathbb{R}^d is said to be N -cyclically monotone if for every cycle $x_1, \dots, x_N, x_{N+1} = x_1$ of points in Ω , one has

$$\sum_{i=1}^N \langle u(x_i), x_i - x_{i+1} \rangle \geq 0. \quad (1)$$

A classical theorem of Rockafellar [10] states that a single-valued map u from Ω to \mathbb{R}^d is N -cyclically monotone for every $N \geq 2$ if and only if

$$u(x) = \nabla \phi(x) \text{ for all } x \in \Omega, \quad (2)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function. On the other hand, a result of E. Krauss [9] yields that u is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$u(x) = \nabla_2 H(x, x) \text{ for all } x \in \Omega, \quad (3)$$

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where H is a concave-convex anti-symmetric Hamiltonian on $\mathbb{R}^d \times \mathbb{R}^d$.

In this paper, we extend the result of Krauss to N -cyclically monotone vector fields, where $N \geq 3$. To formulate variational principles for such vector fields, we let σ to be the cyclic permutation on \mathbb{R}^d , defined by

$$\sigma(x_1, x_2, \dots, x_{N-1}, x_N) = (x_2, x_3, \dots, x_N, x_1),$$

and consider the family of continuous N -cyclically symmetric Hamiltonians on Ω^N , that is

$$\mathcal{H}_N(\Omega) = \{H \in C(\Omega^N); \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) = 0\}$$

We shall say that H is N -cyclically subsymmetric if the equality above is replaced by \leq . We associate to any function H on Ω^N , the following functional on $\Omega \times (\mathbb{R}^d)^{N-1}$,

$$L_H(x, p_1, \dots, p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, \dots, y_{N-1}); y_i \in \Omega \right\}. \quad (4)$$

Note that if Ω is convex and if H is convex in the last $N-1$ variables, then L_H is nothing but the Legendre transform of \tilde{H} with respect to the last $N-1$ variables, where $\tilde{H} = H$ on Ω^N and $= +\infty$ outside of Ω^N . Since $H(x, \dots, x) = 0$ for any $H \in \mathcal{H}_N(\Omega)$, then for any such H , we have for $x \in \Omega$ and $p_1, \dots, p_{N-1} \in \mathbb{R}^d$,

$$L_H(x, p_1, \dots, p_{N-1}) \geq \sum_{i=1}^{N-1} \langle x, p_i \rangle. \quad (5)$$

We also consider the class of σ -invariant probability measures on Ω^N , which are those $\pi \in \mathcal{P}(\Omega^N)$ such that for all $f \in L^1(\Omega^N, d\pi)$, we have

$$\int_{\Omega^N} f(x_1, \dots, x_N) d\pi = \int_{\Omega^N} f(\sigma(x_1, \dots, x_N)) d\pi. \quad (6)$$

We then denote

$$\mathcal{P}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N); \pi \text{ } \sigma\text{-invariant probability on } \Omega^N\}. \quad (7)$$

For a given probability measure μ on Ω , we also consider the class

$$\mathcal{P}_{\text{sym}}^\mu(\Omega^N) = \{\pi \in \mathcal{P}_{\text{sym}}(\Omega^N); \text{proj}_1 \pi = \mu\}, \quad (8)$$

i.e., the set of all $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$ with a given first marginal μ , meaning that

$$\int_{\Omega^N} f(x_1) d\pi(x_1, \dots, x_N) = \int_{\Omega} f(x_1) d\mu(x_1) \text{ for every } f \in L^1(\Omega, \mu). \quad (9)$$

Consider now the set $\mathcal{S}(\Omega)$ of μ -measure preserving transformations on Ω , which can be identified with a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$. We shall also consider the subset of $\mathcal{S}(\Omega)$ consisting of N -involutions, that is

$$\mathcal{S}_N(\Omega) = \{S \in \mathcal{S}(\Omega); S^N = I \text{ a.e.}\}$$

The following lemma deals with those probabilities in $\mathcal{P}_{\text{sym}}^\mu(\Omega^N)$, that are carried by graphs of functions from Ω to Ω^N .

Lemma 1 *Let $S : \Omega \rightarrow \Omega$ be a μ -measurable map, then the following properties are equivalent:*

1. *The image of μ by the map $x \rightarrow (x, Sx, \dots, S^{N-1}x)$ belongs to $\mathcal{P}_{\text{sym}}^\mu(\Omega^N)$.*
2. *S is μ -measure preserving and $S^N(x) = x$ μ -a.e.*
3. *$\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu(x) = 0$ for every $H \in L^1(\Omega^N, \otimes_N d\mu)$ that is N -cyclically symmetric on Ω .*

Proof. It is clear that 1) implies 3) since $\int_{\Omega^N} H(\mathbf{x}) d\pi(\mathbf{x}) = 0$ for any N -cyclically symmetric Hamiltonian H and any $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$.

2) implies 1) is also straightforward since if π is the push-forward of μ by the map $x \rightarrow (x, Sx, \dots, S^{N-1}x)$, where S is μ -measure preserving and $S^N(x) = x$ μ -a.e., and for all $f \in L^1(\Omega^N, d\pi)$, we have

$$\begin{aligned} \int_{\Omega^N} f(x_1, \dots, x_N) d\pi &= \int_{\Omega^N} f(x, Sx, \dots, S^{N-1}x) d\mu(x) = \int_{\Omega^N} f(Sx, S^2x, \dots, S^{N-1}x, S^Nx) d\mu(x) \\ &= \int_{\Omega^N} f(Sx, S^2x, \dots, S^{N-1}x, x) d\mu(x) = \int_{\Omega^N} f(\sigma(x_1, \dots, x_N)) d\pi. \end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that S is measure preserving such that $S^N = I$ a.e., then for every N -symmetric H in $L^1(\Omega^N)$, we have

$$\begin{aligned} \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu &= \int_{\Omega} H(Sx, S^2x, \dots, S^{N-1}x, x) d\mu \\ &= \dots = \int_{\Omega} H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) d\mu. \end{aligned}$$

Since H is N -cyclically symmetric, then

$$H(x, Sx, \dots, S^{N-1}x) + H(Sx, S^2x, \dots, S^{N-1}x, x) + \dots H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) = 0.$$

It follows that $N \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$.

For the reverse implication, assume $\int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$ for every N -cyclically antisymmetric Hamiltonian H . By using that identity with the Hamiltonians

$$H(x_1, x_2, \dots, x_N) = f(x_1) - f(x_i)$$

where f is any continuous function on Ω , one gets that S is measure preserving. Now take the Hamiltonian

$$H(x_1, x_2, \dots, x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that $H \in \mathcal{H}_N(\Omega)$ since it is of the form $H(x_1, \dots, x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1)$. Now apply the identity for such an H to obtain

$$0 = \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = \int_{\Omega} |x - SS^{N-1}x| d\mu.$$

It follows that $S^N = I$ μ a.e., and we are done. \square

2 Characterizations of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d\mu$ is Lebesgue measure dx normalized to be a probability on Ω , and μ can and will then be dropped from all notation. We shall also assume that Ω is convex and that its boundary has measure zero.

Theorem 2 *Let $u : \Omega \rightarrow \mathbb{R}^d$ be a bounded measurable vector field. The following properties are then equivalent:*

1. u is N -cyclically monotone a.e.
2. The infimum of the following Monge-Kantorovich problem

$$\inf \left\{ \int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi(\mathbf{x}); \pi \in \mathcal{P}_{\text{sym}}(\Omega^N) \text{ and } \text{proj}_1 \pi = dx \right\} \quad (10)$$

is equal to zero, and is therefore attained by the push-forward of dx by the map $x \rightarrow (x, x, \dots, x)$.

3. The vector field u is in the polar of $\mathcal{S}_N(\Omega)$, that is

$$\inf\left\{\int_{\Omega}\langle u(x), x - Sx \rangle dx; S \in \mathcal{S}_N(\Omega)\right\} = 0. \quad (11)$$

4. The projection of u on $\mathcal{S}_N(\Omega)$ is the identity map, that is

$$\inf\left\{\int_{\Omega}|u(x) - Sx|^2 dx; S \in \mathcal{S}_N(\Omega)\right\} = \int_{\Omega}|u(x) - x|^2 dx. \quad (12)$$

5. There exists a N -subsymmetric Hamiltonian H which is concave in the first variable, convex in the last $(N - 1)$ variables such that

$$(-u(x), 0, \dots, 0, u(x)) = \partial H(x, x, \dots, x) \quad \text{for } x \in \Omega. \quad (13)$$

Moreover, H is N -symmetric in the following sense

$$H(x_1, x_2, \dots, x_N) + H_{2, \dots, N-1}(x_1, x_2, \dots, x_N) = 0, \quad (14)$$

where $H_{2, \dots, N-1}$ is the concavification of the function $K(\mathbf{x}) = \sum_{i=1}^{N-1} H(\sigma^i \mathbf{x})$ with respect to the last $N - 1$ variables.

6. The following duality holds:

$$\inf\left\{\int_{\Omega} L_H(x, 0, \dots, 0, u(x)) dx; H \in \mathcal{H}_N(\Omega)\right\} = \sup\left\{\int_{\Omega} \langle u(x), Sx \rangle dx; S \in \mathcal{S}_N(\Omega)\right\}$$

and the latter is attained at the identity map.

Remark 3 Note that in the case $N = 2$, $K(\mathbf{x}) = H(x_2, x_1)$ is concave with respect to x_2 , hence $H_2(x_1, x_2) = H(x_2, x_1)$, and (14) becomes

$$H(x_1, x_2) + H(x_2, x_1) = 0,$$

thus H is skew-symmetric, recovering well-known results [9], [4], [7], [8].

Proof: To show that (1) implies (2), it suffices to notice that if π is a σ -invariant probability measure on Ω^N such that $\text{proj}_1 \pi = dx$, then

$$\begin{aligned} \int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi(x_1, \dots, x_N) &= \frac{1}{N} \sum_{i=1}^N \int_{\Omega^N} \langle u(x_i), x_i - x_{i+1} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \int_{\Omega^N} \left(\sum_{i=1}^N \langle u(x_i), x_i - x_{i+1} \rangle \right) d\pi(x_1, \dots, x_N), \end{aligned}$$

which is non-negative whenever u is N -cyclically monotone. On the other hand, if π is the invariant measure obtained by taking the image of $\mu := dx$ by $x \rightarrow (x, \dots, x)$, then

$$\int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi(x_1, \dots, x_N) = 0.$$

To show that (2) implies (3), let S be a μ -measure preserving transformation on Ω such that $S^N = Id$ μ a.e. Then the image π_S of μ by the map

$$x \rightarrow (x, Sx, S^2x, \dots, S^{N-1}x)$$

is σ -invariant, hence

$$\int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi_S(x_1, \dots, x_N) = \int_{\Omega} \langle u(x), x - Sx \rangle dx \geq 0.$$

Again, by taking $S = I$, we get that the infimum is necessarily zero.

That (3) and (4) are equivalent is a standard consequence of the Hahn-Banach theorem.

We now show that (3) implies (1). For that take N points x_1, x_2, \dots, x_N in Ω , and let $R > 0$ be such that $B(x_i, R) \subset \Omega$. Consider the transformation

$$S_R(x) = \begin{cases} x - x_1 + x_2 & \text{for } x \in B(x_1, R) \\ x - x_2 + x_3 & \text{for } x \in B(x_2, R) \\ \dots & \\ x - x_N + x_1 & \text{for } x \in B(x_N, R) \\ x & \text{otherwise} \end{cases}$$

It is easy to see that S_R is a measure preserving transformation and that $S_R^N = Id$. We then have

$$0 \leq \int_{\Omega} \langle u(x), x - S_R x \rangle dx \leq \sum_{i=1}^N \int_{B(x_i, R)} \langle u(x_i), x_{i+1} - x_i \rangle dx.$$

Letting $R \rightarrow 0$, we get from Lebesgue's density theorem, that

$$\frac{1}{|B(x_i, R)|} \int_{B(x_i, R)} \langle u(x), x_i - x_{i+1} \rangle dx \rightarrow \langle u(x_i), x_i - x_{i+1} \rangle,$$

from which follows that T is N -cyclically monotone a.e.

In order to prove that (1) implies (5), we shall need the following lemma.

Lemma 4 *Let $f(x, y) = \langle u(x), x - y \rangle$ and let $f^1(x, y)$ be its convexification in x for fixed y , that is*

$$f^1(x, y) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_k, y) : \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_k = x \right\}. \quad (15)$$

Then, f^1 satisfies the following properties:

1. $f \geq f^1$ on Ω ,
2. f^1 is convex in the first variable and concave with respect to the second,
3. $f^1(x, x) = 0$ for each $x \in \Omega$,
4. f^1 is N -cyclically supersymmetric in the following sense: for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω , we have

$$\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0. \quad (16)$$

Proof. Clearly $f \geq f^1$ and f^1 is convex with respect to x . f^1 is also concave with respect to y , since f itself is concave (actually linear) with respect to y . Note also that $f(x, x) = 0$ and since u is N -cyclically monotone, the function $f(x, y) = \langle u(x), x - y \rangle$ satisfies $\sum_{i=1}^N f(x_i, x_{i+1}) \geq 0$ for any cyclic family $x_1, \dots, x_N, x_{N+1} = x_1$ in Ω .

Fix now $x_1, x_2, \dots, x_N, x_{N+1} = x_1$ in Ω and consider $(x_1^k)_{k=1}^n$ in Ω , and $(\lambda_k)_k$ such $\lambda_k \geq 0$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_1^k = x_1$. For each k , we have

$$f(x_1^k, x_2) + \sum_{i=2}^{N-1} f(x_i, x_{i+1}) + f(x_N, x_1^k) \geq 0,$$

thus, summing over k , and using that f is linear in the second variable, we have

$$\sum_{k=1}^n \lambda_k f(x_1^k, x_2) + \sum_{i=2}^{N-1} f(x_i, x_{i+1}) + f(x_N, x_1) \geq 0,$$

hence

$$f^1(x_1, x_2) + \sum_{i=2}^N f(x_i, x_{i+1}) \geq 0.$$

Let now $\lambda_k \geq 0$, $x_N^k \in \Omega$ be such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k x_N^k = x_N$. We have

$$f^1(x_1, x_2) + \sum_{i=2}^{N-2} f(x_i, x_{i+1}) + f(x_{N-1}, x_N^k) + f(x_N^k, x_1) \geq 0.$$

Multiplying by λ_k , summing over k and using again that f is linear in the second variable, we obtain

$$f^1(x_1, x_2) + \sum_{i=2}^{N-1} f(x_i, x_{i+1}) + \sum_{k=1}^n \lambda_k f(x_N^k, x_1) \geq 0.$$

Hence, by taking the infimum over all possible such choices, we get

$$f^1(x_1, x_2) + \sum_{i=2}^{N-1} f(x_i, x_{i+1}) + f^1(x_N, x_1) \geq 0.$$

Repeat this procedure with $x_{N-1}, x_{N-2}, \dots, x_3$ until we get

$$f^1(x_1, x_2) + f(x_2, x_3) + \sum_{i=3}^N f^1(x_i, x_{i+1}) \geq 0.$$

Note that f^1 is not necessarily linear in the second variable, we cannot apply the same reasoning on x_2 , but we can argue in the following way. Since

$$f(x_2, x_3) \geq -f^1(x_1, x_2) - \sum_{i=3}^N f^1(x_i, x_{i+1}),$$

and since f^1 is concave in the second variable, we have for fixed x_1, x_3, \dots, x_N , the function $x_2 \rightarrow -f^1(x_1, x_2) - \sum_{i=3}^N f^1(x_i, x_{i+1})$ is a convex minorant of $x_2 \rightarrow f(x_2, x_3)$. It follows that

$$f(x_2, x_3) \geq f^1(x_2, x_3) \geq -f^1(x_1, x_2) - \sum_{i=3}^N f^1(x_i, x_{i+1}),$$

which finally implies that $\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0$.

This clearly implies that $f^1(x, x) \geq 0$ for any $x \in \Omega$. On the other hand, since $f^1(x, x) \leq f(x, x) = 0$, we get that $f^1(x, x) = 0$ for all $x \in \Omega$. ■

Proof of 1) implies 5) Let $\psi(x, y) = -f^1(y, x)$ and note that

- (i) $x \rightarrow \psi(x, y)$ is convex,
- (ii) $y \rightarrow \psi(x, y)$ is concave,
- (iii) $\psi(x, y) \geq -f(y, x) = \langle u(y), y - x \rangle$, and

(iv) ψ is N -subsymmetric, meaning that $\psi(x, x) = 0$ and

$$\sum_{i=1}^N \psi(x_i, x_{i+1}) \leq 0 \text{ for all cyclic families } x_1, \dots, x_N, x_{N+1} = x_1 \text{ in } \Omega. \quad (17)$$

Consider now the family \mathcal{H} of functions $H : \Omega^N \rightarrow \mathbb{R}$ such that

1. $H(x_1, x_2, \dots, x_N) \geq \psi(x_N, x_1)$ for all x_1, \dots, x_N in Ω .
2. H is concave in the first variable,
3. H is jointly convex in the last $N - 1$ variables.
4. H is N -subsymmetric in the following sense: $H(x, \dots, x) = 0$ and

$$\sum_{i=1}^N H(\sigma^{i-1} \mathbf{x}) \leq 0 \text{ for all } \mathbf{x} = (x_1, \dots, x_N) \in \Omega^N. \quad (18)$$

Note that $\mathcal{H} \neq \emptyset$ since $H(x_1, x_2, \dots, x_N) := \psi(x_N, x_1)$ belongs to \mathcal{H} . Moreover, by N -subsymmetry, any $H \in \mathcal{H}$ satisfies for all $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$,

$$H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \leq - \sum_{i=1}^{N-1} \psi(x_i, x_{i+1}). \quad (19)$$

This also yields that

$$\langle u(x_1), x_N - x_1 \rangle \leq H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} \langle u(x_{i+1}), x_i - x_{i+1} \rangle, \quad (20)$$

which means that $H(x, x, \dots, x) = 0$ for every $H \in \mathcal{H}$ and any $x \in \Omega$.

On the other hand, it is easy to see that every directed family $(H_i)_i$ in \mathcal{H} has a supremum $H_\infty \in \mathcal{H}$, meaning that \mathcal{H} is a Zorn family, and therefore it has a maximal element H .

Consider now the function

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N},$$

and note that

(i) \bar{H} is N -symmetric, since

$$\bar{H}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N-1} [H(\mathbf{x}) - H(\sigma^i \mathbf{x})],$$

and each $K_i(\mathbf{x}) := H(\mathbf{x}) - H(\sigma^i \mathbf{x})$ is N -symmetric.

(ii) $\bar{H} \geq H$ on Ω^N , since

$$N[\bar{H}(\mathbf{x}) - H(\mathbf{x})] = - \sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \geq 0,$$

because H itself is N -subsymmetric.

The maximality of H would have implied that $H = \bar{H}$ is N -symmetric if only \bar{H} was jointly convex in the last $N - 1$ -variables, but since this is not necessarily the case, we consider for $\mathbf{x} = (x_1, x_2, \dots, x_N)$, the function

$$K(x_1, x_2, \dots, x_N) = K(\mathbf{x}) := - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})),$$

which is already concave in the first variable x_1 . Its convexification in the last $(N-1)$ -variables, that is

$$K^{2,\dots,N}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^n \lambda_i K(x_1, x_2^i, \dots, x_N^i); \lambda_i \geq 0, \sum_{i=1}^n \lambda_i (x_2^i, \dots, x_N^i, 1) = (x_2, \dots, x_N, 1) \right\},$$

is still concave in the first variable, but is now convex in the last $N-1$ variables. Moreover,

$$H \leq K^{2,\dots,N} \leq K = - \sum_{i=1}^{N-1} H \circ \sigma^i. \quad (21)$$

Indeed, $K^{2,\dots,N} \leq K$ from the definition of $K^{2,\dots,N}$, while $H \leq K^{2,\dots,N}$ because $H \leq K$ and H is already convex in the last $(N-1)$ -variables.

It follows that

$$H \leq \frac{(N-1)H + K^{2,\dots,N}}{N} \leq \frac{(N-1)H + K}{N} = \frac{(N-1)H - \sum_{i=1}^{N-1} H \circ \sigma^i}{N} = \bar{H}.$$

The function $H' = \frac{(N-1)H + K^{2,\dots,N}}{N}$ belongs to the family \mathcal{H} and therefore $H = H'$ by the maximality of H .

This finally yields that \bar{H} is N -subsymmetric, that $H(x, x, x) = 0$ for all $x \in \Omega$ and that

$$H(\mathbf{x}) + H_{2,\dots,N}(\mathbf{x}) = 0 \text{ for every } \mathbf{x} \in \Omega^N,$$

where $H_{2,\dots,N} = -K^{2,\dots,N}$, which is the concavification of $\mathbf{x} \rightarrow \sum_{i=1}^{N-1} H(\sigma^i \mathbf{x})$ with respect to the last $N-1$ -variables.

Note now that since for any x_1, \dots, x_N in Ω , $H(x_1, \dots, x_N) \geq \langle u(x_1), x_N - x_1 \rangle$, and $H(x_1, x_1, x_1) = 0$, we have

$$H(x_1, \dots, x_N) - H(x_1, \dots, x_1) \geq \langle u(x_1), x_N - x_1 \rangle.$$

Since H is convex in the last $N-1$ variables, this means that for all $x \in \Omega$, we have

$$(0, \dots, 0, u(x)) \in \partial_{2,3} H(x, x, \dots, x). \quad (22)$$

On the other hand, (20) gives that for any x_1, \dots, x_N in Ω ,

$$\langle u(x_1), x_N - x_1 \rangle \leq H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} \langle u(x_{i+1}), x_i - x_{i+1} \rangle, \quad (23)$$

which means that for any x, y in Ω , we have

$$H(y, x, x, \dots, x) - H(x, x, x) \leq -\langle u(x), x - y \rangle.$$

Since H is concave in the first variable, we deduce that

$$-u(x) \in \nabla_1 H(x, x, x), \quad (24)$$

and consequently

$$(-u(x), 0, \dots, 0, u(x)) \in \partial H(x, x, \dots, x). \quad (25)$$

To prove that 5) implies 6) note that

$$L_H(x, p_1, \dots, p_{N-1}) + H(x, y_1, \dots, y_{N-1}) \geq \sum_{i=1}^{N-1} \langle p_i, y_i \rangle,$$

which yields that

$$\int_{\Omega} [L_H(x, 0, \dots, 0, u(x)) dx + H(x, S^{N-1}x, \dots, Sx)] dx \geq \int_{\Omega} \langle u(x), Sx \rangle.$$

If now $H \in \mathcal{H}_N(\Omega)$ and $S \in \mathcal{S}_N(\Omega)$, we then have $\int_{\Omega} H(x, S^{N-1}x, \dots, Sx) dx = 0$, and therefore

$$\int_{\Omega} L_H(x, 0, \dots, 0, u(x)) dx \geq \int_{\Omega} \langle u(x), Sx \rangle dx.$$

If now H is the N -subsymmetric Hamiltonian H obtained by 5), which is concave in the first variable, convex in the last $N - 1$ variables and such that

$$(-u(x), 0, \dots, 0, u(x)) = \nabla H(x, x, \dots, x) \quad \text{for all } x \in \Omega,$$

then

$$L_H(x, 0, \dots, 0, u(x)) + H(x, x, \dots, x) = \langle u(x), x \rangle \quad \text{for all } x \in \Omega,$$

and therefore $\int_{\Omega} L_H(x, 0, \dots, 0, u(x)) dx = \int_{\Omega} \langle u(x), x \rangle dx$.

Consider now

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

As in the proof of 1) implies 5), we have that $\bar{H} \in \mathcal{H}_N(\Omega)$ and $\bar{H} \geq H$. Since $L_{\bar{H}} \leq L_H$, we have that $\int_{\Omega} L_{\bar{H}}(x, 0, \dots, 0, u(x)) dx = \int_{\Omega} \langle u(x), x \rangle dx$ and (6) is proved.

Finally, note that (6) readily implies (3), which means that u is then N -cyclically monotone.

Remark 5 Note that the sets of measure preserving N -involutions $(\mathcal{S}_N(\Omega))_N$ do not form a nested family, that is $\mathcal{S}_N(\Omega)$ is not necessarily included in $\mathcal{S}_M(\Omega)$, whenever $N \leq M$, unless of course M is a multiple of N . On the other hand, the above theorem shows that their polar sets, that is

$$\mathcal{S}_N(\Omega)^0 = \{u \in L^2(\Omega, \mathbb{R}^d); \int_{\Omega} \langle u(x), x - Sx \rangle dx \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega)\},$$

which coincide with the N -cyclically monotone maps, satisfy for every $N \geq 1$,

$$\mathcal{S}_{N+1}(\Omega)^0 \subset \mathcal{S}_N(\Omega)^0.$$

This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that u is 3-cyclically monotone operator, then for any transformation $S : \Omega \rightarrow \Omega$, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle dx + \int_{\Omega} \langle u(Sx), Sx - S^2x \rangle dx + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle dx \geq 0.$$

If now S is measure preserving, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle dx + \int_{\Omega} \langle u(x), x - Sx \rangle dx + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle dx \geq 0,$$

and if $S^2 = I$, then $\int_{\Omega} \langle u(x), x - Sx \rangle dx \geq 0$, which means that $u \in \mathcal{S}_2(\Omega)^0$. Similarly, one can show that any $(N+1)$ -cyclically monotone operator belongs to $\mathcal{S}_N(\Omega)^0$. In other words, $\mathcal{S}_{N+1}(\Omega)^0 \subset \mathcal{S}_N(\Omega)^0$ for all $N \geq 2$. Note that $\mathcal{S}_1(\Omega)^0 = L^2(\Omega, \mathbb{R}^d)$, while

$$\mathcal{S}(\Omega)^0 = \cap_N \mathcal{S}_N(\Omega)^0 = \{u \in L^2(\Omega, \mathbb{R}^d), u = \nabla \phi \text{ where } \phi \text{ is a convex function in } W^{1,2}(\mathbb{R}^d)\},$$

in view of classical results of Rockafellar [11] and Brenier [1].

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